problems to nullify any improvement in device reliability due to wider spacing However, the experimental data indicate the requirement for circulating cesium may not be necessary Additional data on a cavity emitter patterned after the capillary structure should be obtained in order to determine if ion and electron emission is governed by the random current theory

References

[1] Huber, H and LeBihan, R , "Emetteur capillaire d'ions de cesium: application a des convertisseurs thermioniques," Combustion and Propulsion Panel, AGARD-NATO, Athens, Greece (July 1963)

[2] Dresser, D L , Dunlop, J D , Huber, H , LeBihan, R , Hill, V L , McMurray, N D , and Muszer, K E , "Investigation of capillary emitter thermionic converters," Aeronaut Systems Div TDR63-865 (September 1963)

[3] Dykman, I M , "On the mechanism of activator vapours supply to the outer surface of a porous metal-film cathode," Radiotekhn i Elektron 2, 1525–1257 (1957)

[4] Clausing, P , "Over de Stationmaire Strooming Van Zeer Verdunde Gassen," Physica 9, 65–80 (1929)

[5] Langmuir, I , *Phenomena, Atoms and Molecules* (Philosophical Library, Inc , New York, 1950), pp 322–384

[6] Houston, J M , "Thermionic emission of refractory metals in cesium vapor," Proc Roundtable Discussion, Power Information Center, Rept PIC-ELE-T1 3/3, Appendix F-1, Univ of Penn (June 1961)

[7] Nottingham, W B , "Analysis of typical voltage-current curves," Proc Third Government-Industry Thermionic Roundtable Discussions, Power Information Center Rept PIC-ELE-T1 209/1, pp 1-1–1-9, Univ of Penn (December 1961)

[8] Nottingham, W B , "Sheath and plasma theory of an isothermal diode," Topical Rept for the Thermionic Emitter Materials Res Program, Contract NONR 3563 (00), Thermo Electron Eng Co (1962)

[9] Dresser, D L , "Analysis of capillary converter experimental results," Thermionic Conversion Specialists' Meeting, Gatlinburg, Tenn , October 7–9, 1963 (to be published)

[10] Nottingham, W B , "General theory of the plasma diode energy converter," TR TEE-7002-5, Thermo Electron Eng Co (1960)

[11] Breitwieser, R , "Cesium diode operation in three modes," Rept on 23rd Annual Conference Physical Electronics, Mass Inst Tech , Cambridge, Mass , pp 267–280 (1963)

[12] Johnson, E O , "Studies of externally heated hot cathode arcs," RCA Rev 16, 498–532 (1955)

[13] Hernquist, K G , "Analysis of the arc mode operation of the cesium vapor thermionic energy converter," Proc Inst Electrical Electron Engrs 51, 748–754 (1963)

[14] Wilson, V C and Lawrence, J , "Design and operation of laboratory type parallel plane cesium thermionic converters," Advanced Energy Conversion 2, 335–340 (1962)

[15] Schneider, R T and Dunlop, J D , "Spectroscopic investigation of a thermionic capillary cesium energy converter," Appl Phys Letters 3, 73–75 (1963)

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Effect of Mass Loss on the Transient Response of a Shallow Spherical Sandwich Shell

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The effect of continuous mass loss on the dynamic response of a shallow spherical sandwich shell is considered The material that loses mass is a uniform covering of the shell, and the covering is assumed to experience mass loss at a constant rate as dynamic pressures act on the combined structure An analytical solution is obtained when the pressure exhibits a ramp loading with respect to time The solution shows that, for either high or moderate mass loss rates, the mass loss has a negligible effect on the stress amplitudes in the shell Numerical results for the static shell stresses are presented and discussed

Nomenclature

a = circumferential radius of shell
B = eigenfunction
C = constant of integration
E = modulus of elasticity
G = transverse shear modulus
h = thickness
J_0, Y_0 = zero-order Bessel functions, first and second kind
M,N = resultant moments and forces
q = normal forces on the shell
Q = transverse shear resultant
p = external pressure
r,R = coordinate and spherical radius
t = time
$\mathcal{J}$ = Onoe function, first kind
u = meridional displacement
U = eigenfunction
w = normal displacement
Y = eigenfunction
αh_0 = ablation rate
β = rotation
θ = angular coordinate
λ = separation constant
μ = ablation parameter
ν = Poisson s ratio
ξ = $\xi = r/a$
ρ = mass densities
σ, τ = normal and shear components of stress
ω = frequency

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I Introduction

THE loss of mass from a structure affects dynamic response of the structure through the rate of change of momentum

Whether or not the loss of mass is important depends on how rapid the loss is This paper is concerned with the particular class of time-dependent mass problems which are related to the ablation process and is concerned with determining its effect on the transient response of the supporting structure Because the problem is based on considerations of re-entry vehicles with ablative materials, the specific problem solved in this paper is the response of a shallow spherical sandwich shell supporting a uniform covering layer of material which loses mass at a constant rate At the same time, the structure is acted on by pressures that exhibit a linear rise during a finite time interval, after which the pressure is maintained at a constant value

The results show that the major effect of the mass loss is to alter the response time of the shell Also, for either high or moderate ablation rates, the ablation process has a negligible effect on the stress amplitudes in the backup shell Because of the importance of the static solution, a partial numerical study of the geometric parameters affecting static shell stresses is presented and discussed

Recently, Birnbaum[1] investigated the effect of time-dependent mass on missile bending vibrations He found, in the case of a burning, solid propellant rocket, that the fundamental natural frequency is not more than 45% of the fundamental natural frequency of the corresponding constant mass rocket Despite the extent of the mathematical similarity of the present problem to that of Ref 1, it is interesting to note that a similar effect is not found here The reason for this is discussed in this paper

II Equations of Motion

The shallow spherical sandwich shell is referred to coordinates z, θ, and r as shown in Fig 1 The face layers are each of thickness h_2, the core has a thickness $2h_1$, and the radius to the middle of the core layer is R Considering only symmetrical loads and displacements, the stress equations of equilibrium are readily obtained from Yao's[2] buckling equations for a shallow spherical sandwich shell by neglecting the effect of an initial stress state considered by Yao, and then inserting a term q, representing the normal forces on the shell The stress equations of equilibrium in terms of the notation of this paper are

$$(rN_r)' - N\theta = 0 \tag{1}$$

$$(rQ)' - (r/R)(N \;\; + N\theta) + rq = 0 \tag{2}$$

$$(rM_r)' - M\theta - rQ = 0 \tag{3}$$

In Eqs (1–3), Q denotes the transverse shear resultant N_r, $N\theta$, M , and $M\theta$ are the resultant forces and moments in the r and θ directions acting on any element of the composite shell; a prime denotes a derivative with respect to r The forces and moments in Eqs (1–3) may be expressed in terms of the resultant forces ($N_{\;2},N_{r3},N\theta_2,N\theta_3$) which act on the individual face plates by

$$\left.\begin{aligned} N_r &= N_{r2} + N_{\;3} \qquad & M_r &= h_1(N_{r3} - N_{r2}) \\ N\theta &= N\theta_2 + N\theta_3 \qquad & M\theta &= h_1(N\theta_3 - N\theta_2) \end{aligned}\right\} \tag{4}$$

The sandwich facings are assumed to have negligible flexural rigidity about their own middle surfaces, and the core layer simply transmits shear

Let u, w, and β denote the meridional displacement, the normal displacement, and the rotation, respectively, of any core section of the shell The resultant forces in terms of stresses and displacements are

$$N_{r\binom{2}{3}} = h_2\sigma_{r\binom{2}{3}} = \frac{Eh_2}{1-\nu^2}\left[u' + \nu\frac{u}{r} + (1+\nu)\frac{w}{R} \mp h_1\left(\beta' + \nu\frac{\beta}{r}\right)\right]$$

$$N\theta_{\binom{2}{3}} = h_2\sigma\theta_{\binom{2}{3}} = \frac{Eh_2}{1-\nu^2}\left[\frac{u}{r} + \nu u' + (1+\nu)\frac{w}{R} \mp h_1\left(\frac{\beta}{r} + \nu\beta'\right)\right] \tag{5}$$

$$Q = 2h_1\tau \qquad \tau = G(w' + \beta)$$

where σ and τ are the usual notations for normal and shear components of stress, E the modulus of elasticity of the face plates, ν the Poisson ratio of the face plates, and G the transverse shear modulus of the core For the purposes of this paper, the core material has been assumed to be infinitely stiff in the radial direction Moduli for a less restrictive condition are given in Ref 2

The three contributors to q which will be considered are external pressure, the shell inertia force, and the inertia force of the ablator mass Since the inertia force is equal to the rate of change of linear momentum,† we have

$$q = -p - 2(h_1\rho_1 + h_2\rho_2)\partial^2 w/\partial t^2 - (\partial/\partial t)(h_a\rho_a\partial w/\partial t) \tag{6}$$

where p is the external pressure; t is time; ρ_1, ρ_2, and ρ_a are mass densities of the core, face, and ablator, respectively; and h_a is the thickness of the ablator mass at any instant of time If, as a first approximation, a constant rate of ablation is assumed,[3] then the thickness of the ablator mass at any instant of time is

$$h_a = h_0(1 - \alpha t) \tag{7}$$

where h_0 is the initial ablator thickness, and αh_0 is the ablation rate The restriction $1 \geq 1 - \alpha t \geq 0$ must hold for Eq (7)

Substituting Eqs (4–7) in Eqs (1–3) and introducing the dimensionless variables

$$\xi = r/a \qquad \bar{u} = u/a \qquad y = w/a$$
$$r_E = E/G \qquad r_\rho = \rho_2/\rho_1 \qquad r_h = h_2/h_1$$

where a is the circumferential radius of the shell, we find that the governing equations of motion are

$$\left(\frac{r_E r_h}{(1-\nu^2)}\right)\frac{\partial}{\partial\xi}\left(\frac{\partial\bar{u}}{\partial\xi} + \frac{\bar{u}}{\xi}\right) + \left(\frac{a\, r_E r_h}{R(1-\nu)}\right)\frac{\partial y}{\partial\xi} = 0 \tag{8}$$

$$\frac{\partial^2 y}{\partial\xi^2} + \frac{1}{\xi}\frac{\partial y}{\partial\xi} + \frac{\partial\beta}{\partial\xi} + \frac{\beta}{\xi} - \left(\frac{a}{R}\right)\frac{r_E r_h}{(1-\nu)} \times \left(\frac{\partial\bar{u}}{\partial\xi} + \frac{\bar{u}}{\xi} + 2\frac{a}{R}y\right) = \frac{ap}{2h_1 G} + \left(\frac{1}{a^2}\frac{G}{\rho_1}\right)^{-1}\frac{\partial}{\partial t} \times \left\{\left[1 + r_h r_\rho + \left(\frac{h_0\rho_a}{2h_1\rho_1}\right) - \frac{h_0\rho_a}{2h_1\rho_1}\alpha t\right]\frac{\partial y}{\partial t}\right\} \tag{9}$$

$$\left(\frac{h_2 h_1 r_E}{a^2(1-\nu^2)}\right)\frac{\partial}{\partial\xi}\left(\frac{\partial\beta}{\partial\xi} + \frac{\beta}{\xi}\right) - \left(\frac{\partial y}{\partial\xi} + \beta\right) = 0 \tag{10}$$

In the derivation of the differential equations, the ablator material is assumed to ablate uniformly over the shell Also, Eqs (8) and (10) are equations of constraint on Eq (9) because the meridional and rotatory inertia of the shell and the ablator mass are neglected

A unique solution of Eqs (8–10) requires the specification, at a circumferential edge, of one of each of the following pairs[4]:

$$\left.\begin{matrix} u \text{ or } N \\ w \text{ or } Q \\ \beta \text{ or } M_r \end{matrix}\right\} \tag{11}$$

The clamped edge condition is of practical interest; therefore, at $\xi = 1, \bar{u} = y = \beta = 0$

† The radial momentum of the departing material is not considered In addition, the stiffness of the ablator is neglected; only the mass of the ablator is considered

III Free Vibration before Start of Ablation

Consider free vibration before ablation starts, in which case

$$\alpha = p = 0$$
$$(\bar{u},y,\beta) = (U,Y,B)e^{i\omega t} \tag{12}$$

After inserting Eq (12) into Eqs (8–10) and applying the regularity condition at $\xi = 0$, the free vibration solutions are readily found to be

$$U(\xi) = \frac{\xi}{2} C_1 \left[1 - \frac{n^2}{\bar{n}^2}\frac{(1+\nu)}{(1-\nu)}\right] - C_2(1+\nu)\left(1 + \frac{b}{m^2}\right)\frac{a}{R}\frac{J_1(b^{1/2}\xi)}{b} - C_3(1+\nu)\left(\frac{c}{m^2} - 1\right)\left(\frac{a}{R}\right)\frac{I_1(c^{1/2}\xi)}{c}$$

$$Y(\xi) = \frac{R}{a}\frac{n^2}{\bar{n}^2}\frac{C_1}{(1-\nu)} + C_2\left(1 + \frac{b}{m^2}\right)\frac{J_0(b^{1/2}\xi)}{b^{1/2}} + C_3\left(\frac{c}{m^2} - 1\right)\frac{I_0(c^{1/2}\xi)}{c^{1/2}} \tag{13}$$

$$B(\xi) = C_2 J_1(b^{1/2}\xi) + C_3 I_1(c^{1/2}\xi)$$

where

$$n^2 = r_E r_h (a/R)^2$$
$$m^2 = (1-\nu^2)(a/h_1)^2/(r_E r_h)$$
$$\bar{n}^2 = \lambda^2 - n^2 \qquad \lambda^2 = \omega^2/\omega_0^2$$
$$\begin{Bmatrix} b \\ c \end{Bmatrix} = \frac{\bar{n}^2}{2}\left[\pm 1 + \left(1 + \frac{4m^2}{\bar{n}^2}\right)^{1/2}\right]$$
$$\omega_0^2 = \left(\frac{G}{\rho_1 a^2}\right)\left[1 + r_h r_\rho + \left(\frac{h_0 \rho_a}{2h_1\rho_1}\right)\right]^{-1}$$

When the clamped edge conditions $[U(1) = Y(1) = B(1) = 0]$ are applied to Eq (13), we have the ratios C_1/C_2 and C_3/C_2, and the characteristic equation

$$\left\{1 - \frac{1}{2}\left[1 - \frac{(1-\nu)}{(1+\nu)}\frac{\bar{n}^2}{n^2}\right]\mathcal{J}_1(ic^{1/2})\right\}\left(1 - \frac{c}{m^2}\right)\frac{b}{c} + \left\{1 - \frac{1}{2}\left[1 - \frac{(1-\nu)}{(1+\nu)}\frac{\bar{n}^2}{n^2}\right]\mathcal{J}_1(b^{1/2})\right\}\left(1 + \frac{b}{m^2}\right) = 0 \tag{14}$$

for the determination of λ In Eq (14), $\mathcal{J}_1(b^{1/2})$ and $\mathcal{J}_1(ic^{1/2})$ are the Onoe functions[5] with real and imaginary arguments, respectively, defined in terms of Bessel functions of the first kind as

$$\mathcal{J}_1(x) \equiv xJ_0(x)/J_1(x)$$
$$\mathcal{J}_1(iy) \equiv yI_0(y)/I_1(y)$$

The expressions given by Eqs (13) and (14) will be shown to apply to the ablation period response; only the time part of the solution will change

IV Free Vibration during Ablation

It is convenient to make a transformation to a new time variable τ, as done by Birnbaum,[1] with

$$t = \frac{1}{\mu}\left(1 - \frac{\mu^2}{\omega_0^2}\tau\right)$$
$$\mu = \left(\frac{h_0\rho_a}{2h_1\rho_1}\alpha\right)\left[1 + r_h r_\rho + \frac{h_0\rho_a}{2h_1\rho_1}\right]^{-1} \tag{15}$$

Insertion of $p = 0$ and the transformation (15) in the expression on the right-hand side of Eq (9) will change that expression to

$$(\partial/\partial\tau)(\tau\partial y/\partial\tau)$$

Let us denote the matrix of the spatial operators on the left-hand side of Eqs (8–10) by H The equations of motion can then be written as

$$H\begin{bmatrix}\bar{u}\\ y\\ \beta\end{bmatrix} = \begin{bmatrix}0\\ (\partial/\partial\tau)(\tau\partial y/\partial\tau)\\ 0\end{bmatrix} \tag{16}$$

and can be solved by the method of separation of variables Substituting

$$[\bar{u},y,\beta] = [U(\xi),Y(\xi),B(\xi)]\ g(\tau)$$

into (16) and separating variables, we find

$$H\begin{bmatrix}U\\ Y\\ B\end{bmatrix} + \lambda^2\begin{bmatrix}O\\ Y\\ O\end{bmatrix} = 0 \tag{17}$$

$$(d/d\tau)(\tau dg/d\tau) + \lambda^2 g = 0 \tag{18}$$

where λ is a separation constant

Equation (17) is the same differential equation governing the spatial part of the solution as was found for the free vibration problem without ablation As a result, U, Y, B, and λ are still given by Eqs (13) and (14) Since H is self-adjoint, λ^2 is still real and positive It is important to note that the characteristic equation is independent of ablation In Birnbaum's problem of the bending vibrations of a solid propellant rocket during powered flight, the mass loss rate affects the eigenvalues, and therefore the mode shapes, through the boundary condition relating the normal component of thrust to the end shear of the beam The mass loss rate does not affect the boundary conditions of the re-entering shell considered here Therefore, the eigenvalues and the mode shapes are unaffected The response of the shell during ablation is characterized by a spatial portion of the solution which is independent of mass loss and a time part of the solution which reflects the mass loss The solution to (18) is[1]

$$g(\tau) = RJ_0(2\lambda\tau^{1/2}) + SY_0(2\lambda\tau^{1/2}) \tag{19}$$

where J_0 and Y_0 are the zero-order Bessel functions of the first and second kind, respectively

V Transient Response during Ablation

The forced vibration problem is now considered The differential equations are

$$H\begin{bmatrix}\bar{u}\\ y\\ \beta\end{bmatrix} = \begin{bmatrix}0\\ (\partial/\partial\tau)(\tau\partial y/\partial\tau)\\ 0\end{bmatrix} + \begin{bmatrix}0\\ F(\tau)\\ 0\end{bmatrix} \tag{20}$$

where F $(= ap/2h_1G)$ denotes nondimensional external pressure Pressure will be considered to be uniform over the shell The boundary and initial conditions are

$$\text{at } \xi = 1\text{:}\quad \bar{u} = y = \beta = 0$$
$$\text{at } \tau = \omega_0^2/\mu^2\text{:}\quad \bar{u} = y = \beta = 0; \tag{21}$$
$$\partial\bar{u}/\partial\tau = \partial y/\partial\tau = \partial\beta/\partial\tau = 0$$

We seek a solution of the form

$$\bar{u}(\xi,\tau) = \sum_{j=1}^{\infty} U_j(\xi)g_j(\tau)$$
$$y(\xi,\tau) = \sum_{j=1}^{\infty} Y_j(\xi)g_j(\tau) \tag{22}$$
$$\beta(\xi,\tau) = \sum_{j=1}^{\infty} B_j(\xi)g_j(\tau)$$

where U_j, Y_j, B_j are the eigenfunctions given by (13) for the

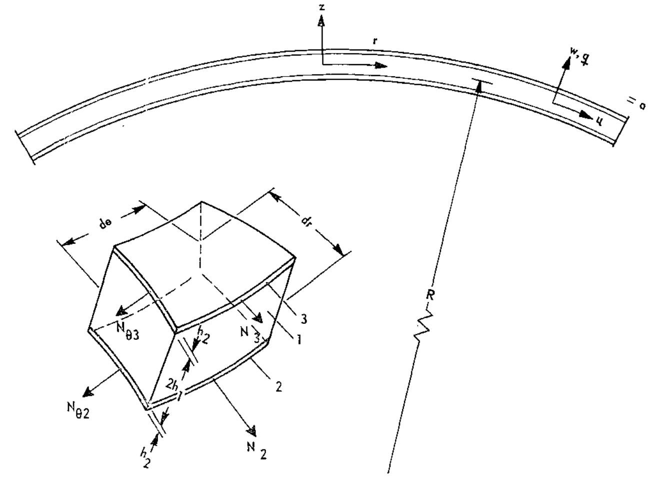


Fig 1 Description of sandwich shell

jth eigenvalue, λ_j, and $C_2^j \equiv 1$ Substituting (22) into (20), we find

$$\sum_{j=1}^{\infty} Y_j \left[-\lambda_j^2 g_j - \left(\frac{d}{d\tau}\right)\left(\frac{\tau d g_j}{d\tau}\right) \right] = F(\tau) \qquad (23)$$

If we multiply (23) by ξY_i, integrate from $\xi = 0$ to 1, and use the orthogonality condition‡

$$\int_0^1 \xi Y_i Y_j d\xi = 0 \qquad i \neq j$$

we obtain

$$(d/d\tau)(\tau d g_j/d\tau) + \lambda_j^2 g_j = P_j(\tau) \qquad (24)$$

for each j, governing the time portion of the solution, in which

$$P_j(\tau) = -F(\tau) A_j$$

$$A_j = \int_0^1 \xi Y_j d\xi \Big/ \int_0^1 \xi Y_j^2 d\xi$$

The general solution of (24) with the initial conditions g_j and $dg_j/d\tau$ specified at $\tau = \tau_1$ is

$$g_j(\tau) = R_j J_0(2\lambda_j \tau^{1/2}) + S_j Y_0(2\lambda_j \tau^{1/2}) + \pi \int_{\tau_1}^{\tau} P_j(\eta)[-Y_0(2\lambda_j \eta^{1/2}) J_0(2\lambda_j \tau^{1/2}) + J_0(2\lambda_j \eta^{1/2}) Y_0(2\lambda_j \tau^{1/2})] d\eta \qquad (25)$$

with

$$\begin{Bmatrix} R_j \\ S_j \end{Bmatrix} = (\mp)\pi\tau_1 \left[g_j(\tau_1) \frac{\lambda_j}{\tau_1^{1/2}} \begin{Bmatrix} Y_1(2\lambda_j \tau_1^{1/2}) \\ J_1(2\lambda_j \tau_1^{1/2}) \end{Bmatrix} + \frac{dg_j}{d\tau}(\tau_1) \begin{Bmatrix} Y_0(2\lambda_j \tau_1^{1/2}) \\ J_0(2\lambda_j \tau_1^{1/2}) \end{Bmatrix} \right]$$

and $\tau \leq \eta \leq \omega_0^2/\mu^2$ The convolution integral in (25) may be verified by direct substitution into (24) The simplest way to obtain the convolution integral is to draw an analogy between (24) and the constant mass single-degree-of-freedom oscillator equation [6] The "velocity" $\tau d\dot{g}_j$, produced by an impulse $P_j d\tau$, is

$$\tau d\dot{g}_j = P_j d\tau$$

The "displacement" dg_j, due to $d\dot{g}_j$, can be computed from (19) Summing the effect of all the impulses, one obtains the convolution integral

The pressures encountered during re-entry are assumed to have a ramp distribution with respect to time, that is,

$$F(t) = \begin{cases} 0 & -\infty < t \leq 0 \\ F_0 t/t_0 & 0 \leq t \leq t_0 \\ F_0 & t_0 \leq t < \infty \end{cases} \qquad (26)$$

‡ See Appendix

Substituting (26) into (25), after transforming from t to τ, and using the initial conditions given by (21), we find g_j to be, for $0 \leq t \leq t_0$,

$$g_j(t) = -F_0 \frac{A_j}{\lambda_j^2} \left\{ \frac{t}{t_0} + \frac{\mu}{\lambda_j^2 \omega_0^2 t_0} + \frac{\pi}{\mu t_0} \left[J_0(2\lambda_j \tau^{1/2}) Y_2\left(\frac{2\lambda_j \omega_0}{\mu}\right) - Y_0(2\lambda_j \tau^{1/2}) J_2\left(\frac{2\lambda_j \omega_0}{\mu}\right) \right] \right\} \qquad (27)$$

and, for $t_0 \leq t \leq t_f$, where t_f is the time at which ablation has stopped,

$$g_j(t) = -\frac{F_0 A_j}{\lambda_j^2} \left\{ \begin{aligned} & 1 + \frac{\pi}{\mu t_0} \left[(1 - \mu t_0) Y_0(2\lambda_j \tau_0^{1/2}) - \frac{\tau_0^{1/2}}{\lambda_j} \left(\frac{\mu}{\omega_0}\right)^2 Y_1(2\lambda_j \tau_0^{1/2}) + Y_2\left(2\lambda_j \frac{\omega_0}{\mu}\right) \right] J_0(2\lambda_j \tau^{1/2}) - \\ & \frac{\pi}{\mu t_0} \left[(1 - \mu t_0) J_0(2\lambda_j \tau_0^{1/2}) - \frac{\tau_0^{1/2}}{\lambda_j} \left(\frac{\mu}{\omega_0}\right)^2 J_1(2\lambda_j \tau_0^{1/2}) + J_2\left(2\lambda_j \frac{\omega_0}{\mu}\right) \right] Y_0(2\lambda_j \tau^{1/2}) \end{aligned} \right\} \qquad (28)$$

In Eq (28), $\tau_0 = (1 - \mu t_0)\omega_0^2/\mu^2$ The displacements with $t > t_f$ revert to the usual expressions using the constant mass oscillator solutions

The numerical evaluation of the solution is carried out for material parameters representative of aluminum honeycomb shells and a phenolic nylon ablative material Therefore, $E = 10 \times 10^6$ psi, $G = 6 \times 10^4$ psi, $\nu = \frac{1}{3}$, $\rho_2 = 0\ 1$ lb/in ^{3}, $\rho_1 = 0\ 00347$ lb/in ^{3}, and $\rho_a = 0\ 0405$ lb/in ^{3} The geometrical parameters chosen are $a/R = 0\ 4$, $h_2/h_1 = 0\ 08$, $a/h_1 = 36\ 12$, $h_0/h_1 = 0\ 65$, and $h_f/h_0 = \frac{1}{4}$, where h_f is the final thickness of the ablator Lastly, the time required for the pressure to rise to its peak value is approximately one-fourth the time required for ablation to be completed As a result, the parameters required for a solution are taken to be $n^2 = 2\ 133$, $m^2 = 86\ 96$, and $\mu t_0 = 0\ 1$ Figures 2a and 2b illustrate the time-dependence of stress at the circumferential boundary of the inner face plate for $t_0/T_1 = 0\ 25$, $\frac{10}{3}$, with T_1 the fundamental period before ablation starts The greater the ratio of rise time to the fundamental period, the more the response tends toward the steady-state static value The effect of halving the time required to ablate (i e , doubling the ablation rate), with constant h_f/h_0, is represented by the $\mu t_0 = 0\ 2$ curve By doing so, we note two prime results: first, the maximum response is only slightly raised, but (second) the response time of the shell is shorter Writing the time part of the response in the form

$$(1 - \mu t)\partial^2 g_j/\partial t^2 - \mu \partial g_j/\partial t + \omega_0 \lambda_j^2 g_j = \omega_0 P_j$$

we see that the first result indicates that the amplitude diminishing effect of the $(1 - \mu t)$ term and the amplitude increasing effect of the $-\mu \partial g_j/\partial t$ term effectively counterbalance each other The second result illustrates that the faster the mass diminishes, the smaller the natural periods become during vibration, and the faster the response of the shell is Last, the limiting case of no ablation ($\mu t_0 = 0$) has been included in Fig 2a to provide a convenient reference curve

In a realistic situation, $t_0/T_1 \gg 1$, and so the response of the shell is a static one The problem considered here has a parallel in the study by Sternberg and Chakravorty[7] on the effect of inertia in the elastic response of a half-space to a thermal input In Ref 7 it was found that, for realistic rise times of thermal input, the problem is a quasi-static one Since ablation is a quasi-static phenomenon, it also does not cause appreciable dynamic stresses

VI Static Solution

The solution for static stresses in the clamped shallow spherical shell subjected to a uniform pressure was computed in terms of its eigenfunction expansion using an IBM 704 digital computer

Some results are shown in Fig 3 for variable a/R and a/h_1, with $\nu = \frac{1}{3}$, $E/G = 166\ 7$, $h_2/h_1 = 0\ 08$ The face stresses plotted in Figs 3a and 3b are the maximum ones at $\xi = 0$ and $\xi = 1$, respectively The eigenvalues are computed by an interval halving procedure once a change in sign is found for the remainder of Eq (14) The eigenvalues are printed when the remainder is of $0(10^{-7})$ or when 75 passes are made at interval halving, whichever comes first The computations are regarded as correct if the fundamental eigenvalue produces a remainder for Eq (14) of $0(10^{-5})$ No results are shown for the upper right-hand corners of Figs 3a and 3b, because the eigenvalues cannot be computed accurately in that region The loss of accuracy is due to the existence of a predominantly membrane state of stress when both a/h_1 and a/R are large

In the regions where the eigenvalues can be computed accurately, the first 20 eigenfunctions are used in the series expansion for stress From a study of the convergence of the series, it is felt that the stresses are accurate to three significant figures This is confirmed for the flat plate ($a/R = 0$) by a comparison with the simple closed-form solution given in Ref 8

Appendix

The desired orthogonality condition for the eigenfunctions of the homogeneous equations of motion is derived by first forming the matrix expression

$$\int_0^1 [U_k Y_k B_k] \left\{ \lambda_j^2 \begin{bmatrix} 0 \\ Y_j \\ 0 \end{bmatrix} = -H \begin{bmatrix} U_j \\ Y_j \\ B_j \end{bmatrix} \right\} \xi d\xi$$

A similar expression can be formed by interchanging the j and k subscripts Subtracting the two expressions, we obtain

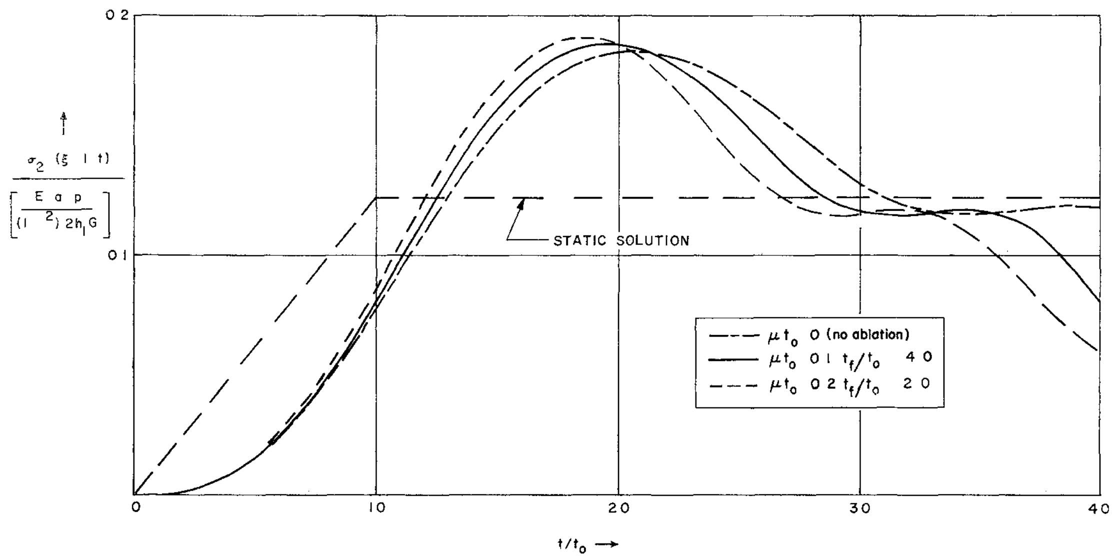


Fig 2a Time-dependence of stress σ_2 at $\xi = 1$, for $t_0/T_1 = 0\ 25$ and different values of dimensionless ablation rate

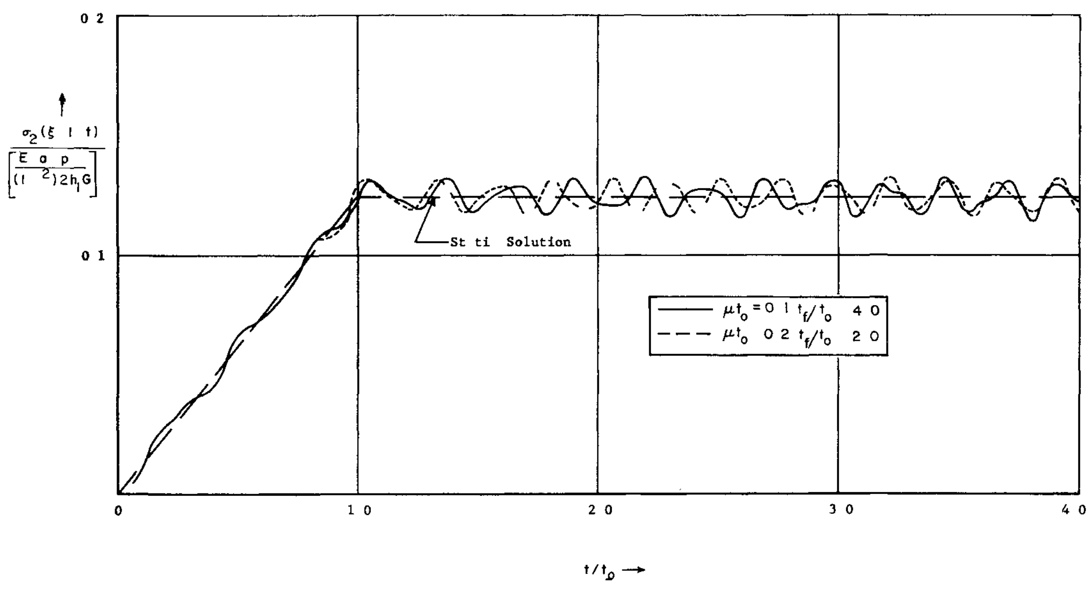
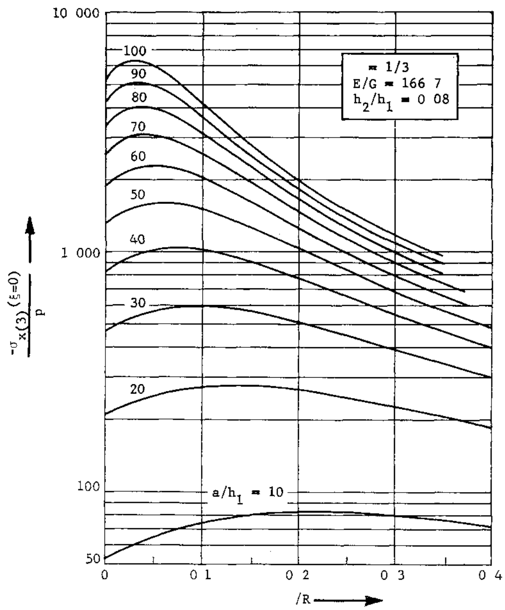
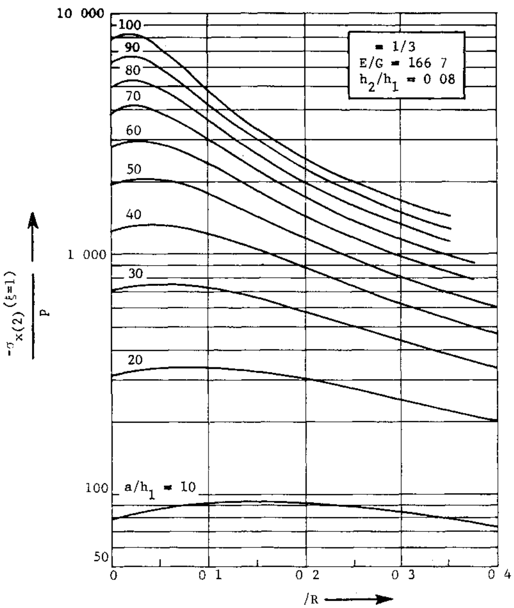


Fig 2b Time-dependence of stress σ_2 at $\xi = 1$, for $t_0/T_1 = \frac{10}{3}$ and different values of dimensionless ablation rate



a)



b)

Fig 3 Static stresses in shell faces

on the left-hand side,

$$(\lambda_j^2 - \lambda_k^2)\int_0^1 \xi Y_j Y_k d\xi$$

After integrating by parts and adding and subtracting both $[\nu B_j B_k/m^2]_0^1$ and $[\nu U_j U_k]_0^1$, we obtain, on the right-hand side,

$$\frac{r_E r_h}{(1-\nu^2)}\left[\left(\frac{\partial U_k}{\partial \xi} + \nu\frac{U_k}{\xi} + \frac{a}{R}(1+\nu)Y_k\right)\xi U_j - \left(\frac{\partial U_j}{\partial \xi} + \nu\frac{U_j}{\xi} + \frac{a}{R}(1+\nu)Y_j\right)\xi U_k\right]_0^1 + \frac{1}{m^2}\left[\left(\frac{\partial B_k}{\partial \xi} + \nu\frac{B_k}{\xi}\right)\xi B_j - \left(\frac{\partial B_j}{\partial \xi} + \nu\frac{B_j}{\xi}\right)\xi B_k\right]_0^1 + \left[\left(\frac{\partial Y_k}{\partial \xi} + B_k\right)\xi Y_j - \left(\frac{\partial Y_j}{\partial \xi} + B_j\right)\xi Y_k\right]_0^1$$

The right-hand side vanishes for any of the eight sets of edge-conditions obtained by equating to zero one member of each of the three pairs in Eq (11), or for any of the associated conditions of elastic support There is no contribution made by the bracketed expressions evaluated at the origin because of the regularity condition Then, if $\lambda_j \neq \lambda_k$,

$$\int_0^1 \xi Y_j Y_k d\xi = 0$$

which is the orthogonality condition

References

[1] Birnbaum S , "Bending vibrations of a perforated grain solid propellant rocket during powered flight," IAS Paper 61-30 (January 1961)

[2] Yao, J C , "Buckling of sandwich sphere under normal pressure," J Aerospace Sci **29**, 254–268 (1962)

[3] Landau, H G , "Heat conduction in a melting solid," Quart Appl Math **8**, 81–94 (1950)

[4] Chu, H N , "On an axisymmetrical problem of the shallow spherical sandwich shell," Martin Co , Denver, Colo , Structures Section Rept (December 1961)

[5] Onoe, M , *Tables of Modified Quotients of Bessel Functions of the First Kind for Real and Imaginary Arguments* (Columbia University Press, New York, 1958)

[6] Lord Rayleigh, *Theory of Sound* (Dover Publications Inc , New York, 1945), 2nd ed , Vol I, p 74

[7] Sternberg, E and Chakravorty, J G , "On inertia effects in a transient thermoelastic problem," J Appl Mech **25**, 503–509 (1959)

[8] Reissner, E "Small bending and stretching of sandwich type shells," NACA TN 1832, pp 36–39 (March 1949)